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SOME SPECIAL SEARCH PROBLEMS

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### SUMMARY

Intuitively, one thinks of a search problem as a puzzle requiring for its solution an efficient technique or algorithm for locating or gaining desirable information about some object. The object may be a physical one or perhaps purely mathematical in character.

Without attempting to define precisely what is meant by a search problem per se, (this we leave to the Logicians) the author illustrates various categories and aspects, e.g., game theoretic, sequential minimax, etc., by means of particular typical problems he and his colleague Selmer Johnson, et al., have solved at RAND.

## SOME SPECIAL SEARCH PROBLEMS

O. Gross

### Fellow Researchers:

If one were to make some sort of snap judgment on the basis of the title of the talk I'm about to present, perhaps the most obvious would be that my colleague Dr. Johnson and I are qualified specialists in the field of search problems (so called), and indeed, the more special the problem the more apt we are to solve it. Nothing could be further from the truth. As a matter of fact, some of the problems we have solved are quite general in applicability, particularly so in those instances in which the principle of optimality of the theory of Dynamic Programming is used as a tool to facilitate solution. Another counterexample to this "specialist" theory, at least insofar as the speaker is concerned, is furnished by his complete inability to solve geographical search problems of the most elementary and special character. In fact, had it not been for the kind graces of a colleague, who furnished his transportation here, he never would have been able to find this place, and you would have been spared the ordeal of listening to his babbling. However, the very fact that this presentation

is being given on Columbus Day lends an aura of appropriateness to the title.

Before we begin discussing our various examples, a philosophical digression is in order. Let us ask ourselves the following question: "What constitutes a search problem?" I, for one, profess not to know the answer to this question, nor do I propose to formulate a precise definition. As a matter of fact, my colleague Selmer Johnson and I have been knocking our heads together in a vain attempt to do so. Perhaps the reason for our lack of success stems principally from the great variety of aspects and kinds of search techniques, schizophrenia and other kinds of limited memory constraints on the search instruments, uncertainty, unreliability of information, animate adversaries attempting to withhold information, etc., that can be involved in a search problem. Apparently, then, any definition capable of encompassing all or any of these aspects would be either too special or too general.

Consequently, it seems advisable to leave the formulation of a definition on the shoulders of the Logician and content ourselves with our intuitive notion as an acceptable answer. (The Logician, of course, when posed the question, may wave his hands in a despairing gesture and exclaim, "Why, any mathematical problem is a search problem—and, indeed, so is

Mathematics itself!")

Well, enough for this digression.

Intuitively, however, one can think of a search problem as a puzzle requiring for its solution an efficient technique or algorithm for locating or gaining desirable information about some object. The object may be a physical one, or perhaps, purely mathematical in character. Lacking a precise definition, we have decided to classify search problems into two broad categories; namely, game theoretic and non-game theoretic. The only reason for this type of breakdown, aside from the speaker's taste for game theory, is that the problems studied in the early days of RAND were of the former type, specifically, two person zero sum games. Unfortunately, due to certain directives, I am not at liberty to discuss any of these.

I suppose it's time to illustrate these remarks by our first example, which can be placed in either category.

With R. Bellman's permission, I shall quote from his RAND paper P-1580 entitled "The Research Frontier":

"Problem 4. Lost in the woods

"Suppose that we parachute into a large forest whose general shape and dimensions we know precisely. If, however, we do not know where we are in the forest, and can locate no point exactly until we reach the boundary, how do we get out of the forest as quickly as possible?"

Admittedly the foregoing question is at best an incomplete formulation of a class of mathematical problems, but, perhaps, purposely so in order to provoke effort on the part of the speaker and others toward the formulation and solution of meaningful interpretations. Now, two different interpretations are readily suggested:

Interpretation No. 1 Game Theoretic

A pure strategy for player I is a covering of the origin of the Euclidean plane by a closed figure of given shape and size.

A pure strategy for player II (who is unaware of the result of I's choice) consists of any pure continuous decision method leading him to a directed rectifiable arc starting at the origin and terminating at the first point it touches on the boundary of the figure placed by player I.

The payoff from II to I is then the length of the path thus obtained.

Interpretation No. 2 Minimax

Find a shortest directed plane rectifiable arc with the property that if the origin of the arc is covered in any way by a given closed plane figure, some point of the arc lies on the boundary of the figure. (We remark that certain figures of infinite extent do not apply here; for example, a half-space. On the other hand, an infinite strip of unit width does.)

As our example, consider the case in which the forest is a circular disc of unit diameter. As far as the game theoretic interpretation is concerned, it is a trivial exercise to verify that:

The unique optimal strategy for player I is to center the circle at the origin.

There are many optimal strategies for player II (the searcher) but all of them (in view of player I's strategy) involve mixing on an initial direction and heading out thereafter in a straight line. For example, mixing equally on a choice of north and south is optimal. Choosing a direction  $\theta$  from the uniform distribution over  $[0, 2\pi]$  is also optimal.

The value of the game is, of course, the radius of the circle, i.e.,  $1/2$ .

The minimax solution for the circular forest is unique and is simply a straight line segment of unit length. This also is a trivial exercise. You observe, however, that the solutions for the two interpretations of the circular forest problem are essentially different.

For other examples and solutions of the forest problem, we refer you to "A Search Problem due to Bellman," RAND research memorandum RM-1603 by O. Gross.

So much for the forest problem.

Our next three examples are non-game theoretic, and, in fact, devoid of random elements. They do, however, display the minimax character mentioned earlier.

I shall quote from a paper by Lester Ford, Jr. and Selmer Johnson. The paper is entitled "A Tournament Problem" and appeared in the May 1959 issue of the American Math. Monthly.

"In his book, 'Mathematical Snapshots', Steinhaus discusses the problem of ranking  $n$  objects according to some transitive characteristic, by means of successive pairwise comparisons. In this paper we shall adopt the terminology of a tennis tournament by  $n$  players. The problem may be briefly stated: 'What is the smallest number of matches that will always suffice to rank all  $n$  players?'

"Steinhaus proposes an inductive method whereby, the first  $k$  players having been ranked, the  $k + 1$ -st player is matched against a median player in the first  $k$  and by a 'halving' process is finally ranked into this chain. Then the  $(k + 2)$ -nd player is ranked into the new chain of  $k + 1$  players in the same manner."

Steinhaus then asserts, "It has not been proved that there is no shorter proceeding possible, but we rather think it to be true."

Steinhaus was wrong in his conjecture, in fact for all  $n > 4$ , as the results of Ford and Johnson show. Their technique

is quite ingenious, but I would rather refer you to their paper for the details. Suffice it to say that, although the authors cannot honestly assert that their method is optimal for all  $n$  (this is unknown) they have verified optimality up to  $n = 11$  and for  $n = 20$  and 21 as well. At any rate, their bound is an improvement over Steinhaus's for comparatively small  $n$ , and even asymptotically ...

Our next example deals with a result obtained independently by Selmer Johnson and J. Kiefer. Although Kiefer's work antedates Johnson's, it is somewhat abstract by comparison, and lacks the straightforward simplicity of Johnson's paper.

Consequently, I shall quote from his RAND paper P-856, which bears the resounding title "Best Exploration for Maximum is Fibonaccian".

"Definition: A function  $f(x)$  is unimodal if there is an  $x_0$  such that  $f$  is either strictly increasing for  $x \leq x_0$  and strictly decreasing for  $x > x_0$ , or else strictly increasing for  $x < x_0$  and strictly decreasing for  $x \geq x_0$ .

"For example, concave functions are unimodal."

The main results of this paper read as follows:

Theorem 1. Let  $y = f(x)$  be any unimodal function defined on an interval  $0 \leq x \leq L_n$ . Let  $F_n$  = the supremum of all  $L_n$  with the property that we can always locate the maximum of  $f(x)$  to within a unit length sub-interval by appropriate calculation of  $n$  value of the function.

Then  $F_n$  is the n-th Fibonacci number, that is  $F_0 = F_1 = 1$  and  $F_{n+2} = F_{n+1} + F_n$ .

Theorem 2. (The discrete case) Let  $y = f(x)$  be any unimodal function defined on a discrete set of  $H_n$  points. Let  $K_n = \text{maximum of all } H_n \text{ such that the function's maximum can be identified in } n \text{ observations.}$

$$\text{Then } K_n = F_{n+1} - 1, \quad n \geq 1.$$

Those of you who are interested in pursuing these results further are referred to Johnson's paper.

I should like to remark, however, that extensions of this type of minimax result to the important multidimensional cases seem extremely difficult to obtain and, to our knowledge, do not exist in the literature.

The next example we shall discuss is gleaned from a joint paper by Johnson and me, entitled "Sequential Minimax Search for a Zero of a Convex Function". This paper originated at RAND and later appeared in a somewhat digested form in the January 1959 issue of MTAC.

The problem we proposed to solve can be stated as follows:

We know initially a positive and a negative value of a function at two given points. The function is continuous and convex and is otherwise unknown but computable. Given a positive integer  $n$ , how can we locate the bracketed zero of

the function to within an interval of minimal guaranteeable length in n evaluations of the function?

Unfortunately, the algorithm which expresses the solution of this problem is somewhat complicated, but not inordinately so. The important point, however, which we wish to bring out as the salient feature of this problem and which is to be shared with others of its kind, is the following:

. The problem requires using an algorithm to find an algorithm which solves the problem. To be more specific, the information structure of the problem, together with dominance considerations, enabled the proposers to invoke the principle of optimality of the theory of Dynamic Programming. A functional equation was thus set up. The recursive solution of the functional equation was accomplished numerically on an electronic digital computer for  $n = 1, 2, 3, 4$ . Reference to the tabular data thus obtained then constitutes part of the algorithm described as the solution to the problem.

Of course, the recursive evaluation of the solution of the functional equation posed a search problem in its own right. But, since it was not a critical one, the proposers did not require any high degree of efficiency in its solution. (The Logician might blurt out at this point, "I told you so!" ) ...

We shall now turn our attention to a search game proposed by Professor David Gale of Brown University. He and I subsequently solved this game, although the solution and its verification are at present unavailable in any published form.

The game can be interpreted as a game over function space and is described thus:

Player I, the maximizer, is in possession of one unit of volume of some valuable product. There are two holes 1 and 2 of depths  $d_1$  and  $d_2$ , respectively and of unit cross-section. To play the game, I levels his product in the bottom of the holes, i.e., he either hides all of it in one of the holes or divides it into two closed parts  $x_1, x_2$ , with  $x_1 > 0, x_1 + x_2 = 1$ , and hides  $x_1$  in hole 1 and  $x_2$  in hole 2. The holes are now filled to ground level with opaque dirt, whereupon player II, the minimizer, enters the scene. II knows, of course, the locations and corresponding depths of the holes, but aside from the rules of the game is ignorant of I's choice. It is now up to II to uncover the loot by removing dirt from the holes in level layers according to some digging scheme. Player II stops digging as soon as he sees pay dirt, whereupon he pays to player I an amount proportional to the volume of dirt he has removed from the holes.

The most interesting case is, apparently, the one in which both holes have depths exceeding unity. Gale and I were able to verify that, for this case, the following describes a solution to the game:

An optimal strategy for player I consists of his behaving as follows:

He hides his entire unit in hole 1 or in hole 2, or, he partitions his unit in the two holes according to the uniform

distribution over the unit interval; the respective probabilities of these three events being proportional to the three numbers,  $d_1 - 1$ ,  $d_2 - 1$  and 1.

Player II (the searcher) has an optimal strategy involving mixing on four paths; namely:

- (A) Dig hole 1 to bottom and then switch to hole 2;
- (B) Dig hole 2 to bottom and then switch to hole 1;
- (C) Dig hole 1 to depth  $d_1 - 1$  and switch; or, finally,
- (D) Dig hole 2 to depth  $d_2 - 1$  and switch.

The respective probabilities of these 4 digging schemes are proportional to the four numbers  $d_1$ ,  $d_2$ ,  $d_1 - 1$  and  $d_2 - 1$ .

The value of the game is expressible as a rational function of the two depths.

We dispense with this oddity by remarking that the searcher in this game has more than one optimal strategy. The uniqueness question for the hider has not been resolved.

You will undoubtedly be pleased to learn that I have only two more examples left to discuss. Both of them are non-game theoretic. They are, however, statistical in nature, in that the random elements are outside the control of the searcher who, moreover, is assumed to be a perfectly reliable observer. The a priori distributions of the random elements are assumed known and unalterable.

Perhaps the distinguishing feature of this type of search problem is that the searcher need not resort to using any

random devices of his own making, for conducting the search. No such device could possibly enhance his expectation. (Incidentally, this last remark is not intended to squelch any Monte Carlo enthusiasts present).

The simpler of the two examples is described in a RAND memorandum RM-1652, March 1956 by Johnson. "Optimal Sequential Testing" is the title. With your leave I shall quote from the Introduction: (TV servicemen kindly take note.)

"A complicated machine, for example, a radar set, which consists of many components each having many parts, is not working because of failure of some of its components. In what sequence should its components and parts be tested and repaired in order to minimize the expected delay time before it is working again?"

Johnson solves this problem under various realistic assumptions about the machine composition, testing times, etc.

The only comment I shall make, however, is that, although quite a few parameters enter into the various models described in the paper, the optimal test sequence in each case is calculated by an absurdly simple ordering. Save for the item to be tested last, the items are ordered according to the algebraic ordering of the values of simple rational functions of the parameters involved. The item to be placed last is determined by a minor calculation and shifted to the end.

For the exact details, I refer you to Johnson's paper...

As our final example of a search problem, or rather a class of such, I recommend a paper which appeared in the Annals of Mathematical Statistics, December 1956. The paper has a trio of authors, R. N. Bradt, Selmer Johnson, and Sam Karlin, and is entitled "On Sequential Designs for Maximizing the Sum of n Observations".

The main theorems of this work are, in effect, statements about optimal sequential designs for various generalizations and concomitant cases of the so called "2-armed bandit problem". I shan't bore you with any mathematical details, but merely give a quotation, as usual:

"The type of problem known as the 'two-armed bandit' is a special case of the preceding. In its 'classical' formulation (whence the name), we have a slot machine with two arms, an x-arm and a y-arm. When either arm is pulled, the machine pays off either one unit or nothing; and the probability of winning with one arm is p and the other, q. A priori it is unknown which is which, but the probability  $\xi$  that it is the x-arm which has probability p of success is assumed known. One is allowed n plays, and a sequential design, or strategy, is desired which will maximize the expected winnings.

"It has been conjectured for this problem that the optimal strategy is  $S_1$ : (namely) on each

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play choose the arm having, at that time, the maximum expected probability of paying off, i.e., play each time as though there were but one play remaining. This conjecture has been verified to hold for  $n \leq 8$ ."

It has been shown, by the way, that the same conjecture as the foregoing is false when applied to generalizations of the two-armed bandit problem, i.e.,  $S_1$  is, in general, not optimal.

At any rate, I suggest you read the paper very carefully before considering a trip to "Vegas".

Before closing, I should like to make a philosophical comment about statistical type non-game theoretic search problems like the two preceding. In certain instances statisticians have been unable to assign a priori distributions when faced with "real world" problems of this kind. In the fairly recent past when game theory was the "coming thing", it had been the tendency for some statisticians to quantify out the unknown distributions by resorting to playing a game against Nature and thus to obtain a certain type of distribution-free result; they would tacitly recommend, for example, recourse to tables of random numbers, in playing some of their games. The implication is obvious. Nature is their enemy; hence they should be careful to avoid using RAND's table of random digits, which were generated by Nature.

I do not know whether this kind of hedging is still in

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vogue\*. At best it is contrary to my religious beliefs . . .

Although this talk has been too long already, we are forced to admit that we have just barely touched upon but a few of the manifold elements that can appropriately be called "ingredients" of a search problem. We hope, however, that you have gleaned some idea of our intuitive notion thereof, by way of the melange of examples presented.

Thank you for your kind attention.

\*A poll of at least one statistician indicates that it is not.

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REFERENCES

1. Bellman, R., "The Research Frontier," RAND P-1580, December 1958.
2. Gross, O., "A Search Problem Due to Bellman," The RAND Corporation Research Memorandum RM-1603, September 1, 1955.
3. Ford, Jr., Lester and Selmer Johnson "A Tournament Problem" The American Math. Monthly, May 1959.
4. Johnson, S. M., "Best Exploration for Maximum is Fibonacci," The RAND Corporation, P-856, May 1956.
5. Gross, O., and S. M. Johnson, "Sequential Minimax Search for a Zero of a Convex Function," Math. Tables and other Aids to Computation, January 1959.
6. Johnson, S. M., "Optimal Sequential Testing," The RAND Corporation, Research Memorandum RM-1652, March 1956.
7. Bradt, R. N., S. M. Johnson, S. Karlin, "On Sequential Designs for Maximizing the Sum of n Observations," Annals of Math. Stat., December 1956.